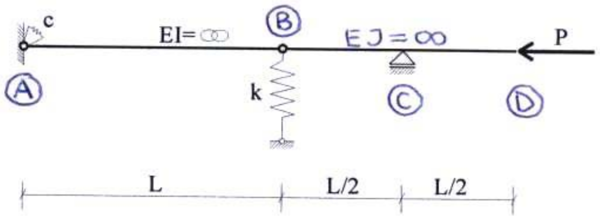
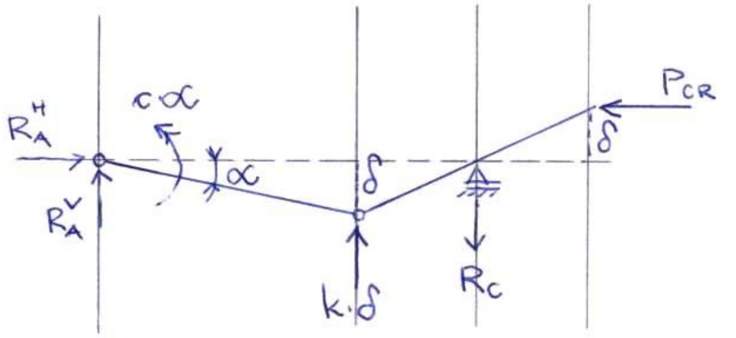
*SVEUČILIŠTE JOSIPA JURJA STROSSMAYERA U OSIJEKU*
*GRAĐEVINSKI FAKULTET OSIJEK*
*Zavod za tehničku mehaniku*
***STABILNOST KONSTRUKCIJA***

*Riješeni primjer I. kolokvija*

1. Odredite kritičnu silu sustava na slici:
   a. pristupom statičke ravnoteže
   b. energetskim pristupom.




a) statička ravnoteža:




$$\alpha \doteq \frac{\delta}{L}$$

$$\Sigma M_A = \emptyset = P\cdot\delta - R_C\cdot 1{,}5L + k\cdot\delta\cdot L + c\cdot\alpha$$

$$R_C = \frac{P\cdot\delta + kL\delta + c\cdot\alpha}{1{,}5L}$$

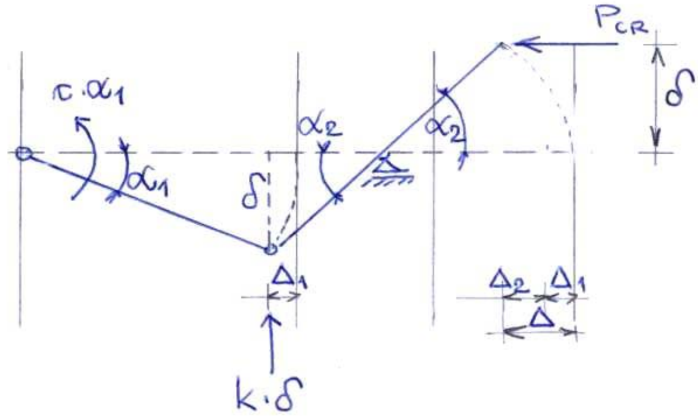
$$\Sigma M_B^D = \emptyset = P\cdot 2\delta - R_C\cdot 0{,}5L$$

$$2P\delta = \frac{P\cdot\delta + kL\delta + c\cdot\alpha}{1{,}5\cancel{L}}\cdot\frac{1}{2}\cancel{L}$$

$$6P\cancel{\delta} = P\cdot\cancel{\delta} + kL\cancel{\delta} + c\frac{\cancel{\delta}}{L} \quad /:\delta$$

$$5P = kL + \frac{c}{L} \Rightarrow P_{CR} = \frac{kL^2 + c}{5L}$$

b) Energetski pristup




$$\alpha_1 \doteq \frac{\delta}{L}$$

$$\alpha_2 \doteq \frac{\delta}{\frac{L}{2}} = \frac{2\delta}{L}$$

$$\Delta_1 \doteq \frac{1}{2} L \cdot \alpha_1^2 = \frac{1}{2} \cdot L \cdot \frac{\delta^2}{L^2} = \frac{\delta^2}{2L}$$

$$\Delta_2 \doteq \frac{1}{2} \cdot L \cdot \alpha_2^2 = \frac{L}{\cancel{2}} \cdot \frac{\overset{2}{\cancel{4}}\delta^2}{L^{\cancel{2}}} = \frac{2\delta^2}{L}$$

– ukupna potencijalna energija: $\Pi = V + U$

$$V = -P \cdot \Delta = -P \cdot (\Delta_1 + \Delta_2) = -P \cdot \left(\frac{\delta^2}{2L} + \frac{2\delta^2}{L}\right) = -\frac{5P\delta^2}{2L}$$

$$U = \frac{1}{2} c \cdot \alpha_1^2 + \frac{1}{2} k \cdot \delta^2 = \frac{1}{2} c \cdot \frac{\delta^2}{L^2} + \frac{1}{2} k\delta^2$$

– za min. potencijala:

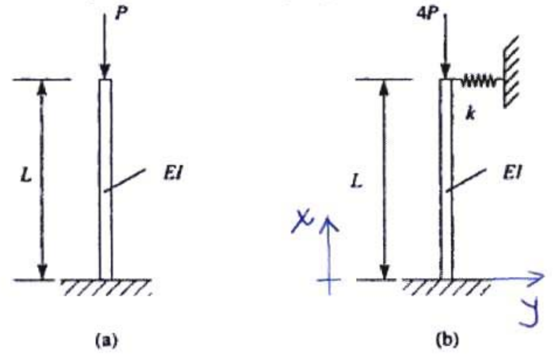
$$\delta\Pi = \frac{\partial \Pi}{\partial \delta}\, \delta\delta = 0 \quad \Rightarrow \frac{\partial \Pi}{\partial \delta} = 0$$

$$-\frac{5P\cancel{\delta}}{L} + \frac{c \cdot \cancel{\delta}}{L^2} + k \cdot \cancel{\delta} = 0 \quad / : \delta \,/\, \cdot L^2$$

$$5PL = c + kL^2$$

$$P_{cr} = \frac{kL^2 + c}{5L}$$

2. Jednoliki stup duljine $L$ i krutosti na savijanje $EI$ upet je s donje a slobodan s gornje strane. Dimenzioniran je tako da može preuzeti kritičnu silu $P$ (slika a). Naknadno je postavljen zahtjev za povećanom nosivosti što je izvedeno dodavanjem poprečne opruge na slobodnom kraju štapa (slika b). Odredite potrebnu krutost opruge, $k$, tako da kritična sila bude $4P$.




Rubni uvjeti:

$x=0: \quad y=0 \quad (1)$

$\qquad\quad y'=0 \quad (2)$

$x=L: \quad M=0 \quad (3)$

$\qquad\quad T=k\cdot y \quad (4)$

$$(1): \quad y(x=0) = B + D = 0 \quad \Rightarrow \quad D = -B$$

$$(2): \quad y'(x=0) = \alpha A + C = 0 \quad \Rightarrow \quad C = -\alpha A$$

$$(3): \quad M(x=L) = -EJ\, y''(x=L) = -EJ\cdot(-\alpha^2 A \sin\alpha L - \alpha^2 B\cos\alpha L) = 0$$

$$A\sin\alpha L + B\cos\alpha L = 0$$

$$(4): \quad T(x=L) = -EJ\left[y'''(x=L) + \alpha^2 y'(x=L)\right] = -k\cdot y(x=L)$$

$$EJ(\cancel{-\alpha^3 A\cos\alpha L} + \cancel{\alpha^3 B\sin\alpha L} + \cancel{\alpha^3 A\cos\alpha L} - \cancel{\alpha^3 B\sin\alpha L} + \alpha^2 C)$$

$$= k\cdot(A\sin\alpha L + B\cos\alpha L + CL + D)$$

$$A\sin\alpha L + B\cos\alpha L = 0$$

$$kA\sin\alpha L + kB\cos\alpha L - k\alpha AL - kB + EJ\alpha^3 A = 0$$

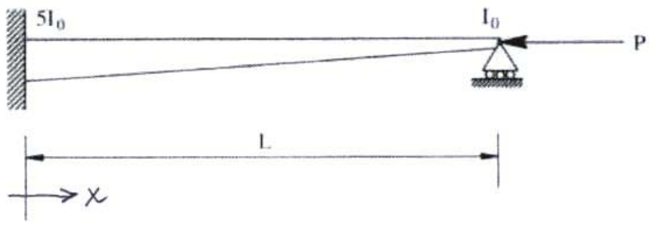
$$\begin{bmatrix} \sin\alpha L & \cos\alpha L \\ k\sin\alpha L - k\alpha L + EJ\alpha^3 & k\cos\alpha L - k \end{bmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix} = \{0\}$$

$$\cancel{k\sin\alpha L\cos\alpha L} - k\sin\alpha L - \cancel{k\sin\alpha L\cos\alpha L} + k\alpha L\cos\alpha L - EJ\alpha^3\cos\alpha L = 0 \qquad /:\cos\alpha L$$

$$k(\alpha L - \mathrm{tg}\,\alpha L) = EJ\alpha^3 \quad , \quad \alpha^2 = \frac{4P}{EJ}$$

$$k = \frac{4P\alpha}{\alpha L - \mathrm{tg}\,\alpha L}$$

3. Odredite kritičnu silu konzole duljine $\boldsymbol{L}$ promjenjive krutosti pomoću Rayleigh-evog kvocijenta.




- FUNKCIJA PROMJENE MOMENTA TROMOSTI DUŽ OSI ŠTAPA:

$$I(x) = I_1 + \frac{x}{L}(I_0 - I_1)$$

za $I_1 = 5I_0$ : $\quad I(x) = I_0\left(5 - \frac{4x}{L}\right)$

- PRIJEDLOG FUNKCIJE PROGIBNE LINIJE, $y(x)$

geometrijski rubni uvjeti konstrukcije:

$$y(\emptyset) = y'(\emptyset) = \emptyset \quad i \quad y(L) = \emptyset \quad \Rightarrow \quad m = 3$$

$$n\,(\text{stupanj polinoma}) = m + 1 = 4$$

dodatni fizikalni rubni uvjet: $\quad y''(L) = \emptyset$

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

$$y(x=\emptyset) = \emptyset \quad \Rightarrow \quad a_0 = \emptyset$$

$$y'(x=\emptyset) = \emptyset \quad \Rightarrow \quad a_1 = \emptyset$$

$$y(x=L) = a_2 L^2 + a_3 L^3 + a_4 L^4 = \emptyset$$

$$y''(x=L) = 2a_2 + 6a_3 L + 12 a_4 L^2 = \emptyset$$

pretpostavimo da je $a_4 L^4 = a$ (nova konstanta)

tada iz posljednje dvije jednadžbe slijedi:

$$a_2 = \frac{3}{2} \cdot \frac{a}{L^2} \quad i \quad a_3 = -\frac{5}{2} \cdot \frac{a}{L^3}$$

- KONAČNO JE:

$$y(x) = a\left[\frac{3}{2}\left(\frac{x}{L}\right)^2 - \frac{5}{2}\left(\frac{x}{L}\right)^3 + \left(\frac{x}{L}\right)^4\right]$$

RAYLEIGH:
$$P_{cr} = \frac{\int_0^L EJ\left(y''(x)\right)^2 dx}{\int_0^L \left(y'(x)\right)^2 dx}$$

$$y'(x) = a\left[\frac{3x}{L^2} - \frac{15x^2}{2L^3} + \frac{4x^3}{L^4}\right], \quad y''(x) = a\left[\frac{3}{L^2} - \frac{15x}{L^3} + \frac{12x^2}{L^4}\right]$$

$$\int_0^L EJ(x)\cdot\left(y''(x)\right)^2 dx = \int_0^L EJ_0\left(5 - \frac{4x}{L}\right)\left[a\left(\frac{3}{L^2} - \frac{15x}{L^3} + \frac{12x^2}{L^4}\right)\right]^2 dx =$$

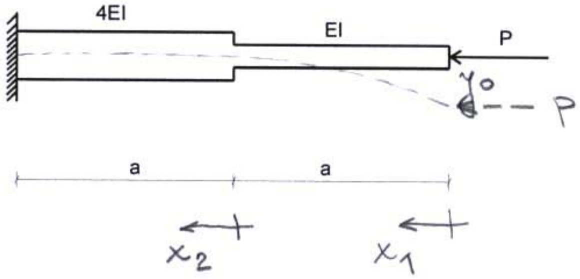
$$= \frac{6EJ_0 a^2}{L^3}$$

$$\int_0^L \left(y'(x)\right)^2 dx = \int_0^L \left[a\left(\frac{3x}{L^2} - \frac{15x^2}{2L^3} + \frac{4x^3}{L^4}\right)\right]^2 dx = \frac{0{,}086\, a^2}{L}$$

$$P_{cr} = \frac{\frac{6EJ_0 a^2}{L^3}}{\frac{0{,}086\, a^2}{L}} = \frac{69{,}8\, EJ_0}{L^2} \doteq \frac{7{,}1\,\pi^2 EJ_0}{L^2}$$

$$= \frac{\pi^2 EJ_0}{(0{,}376L)^2}$$

4. Odredite kritičnu silu konzole duljine ***2a*** promjenjive krutosti, metodom početnih parametara.




$$\alpha_1^2 = \frac{P}{EJ} \equiv \alpha^2 \quad ; \quad \alpha_2^2 = \frac{P}{4EJ} = 0{,}5\alpha$$

$$0 \le x_1 \le a$$

$$y(x_1 = a) = y_a = y_0 + y_0' \frac{\sin \alpha_1 a}{\alpha_1}$$

$$y'(x_1 = a) = y_a' = y_0' \cos \alpha_1 a$$

$$M(x_1 = a) = M_a = \alpha_1 EJ\, y_0' \sin \alpha_1 a$$

$$T(x_1 = a) = T_0 = \emptyset$$

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$$0 \le x_2 \le a$$

$$y(x_2 = a) = y_a + y_a' \frac{\sin \alpha_2 a}{\alpha_2} - M_a \frac{1 - \cos \alpha_2 a}{\alpha_2^2 \cdot 4EJ} = \emptyset$$

$$y'(x_2 = a) = y_a' \cos \alpha_2 a - M_a \frac{\sin \alpha_2 a}{\alpha_2 \cdot 4EJ} = \emptyset$$

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$$y(x_2 = a) = y_0 + y_0' \frac{\sin \alpha a}{\alpha} + y_0' \cos \alpha a \cdot \frac{\sin 0{,}5\alpha a}{0{,}5\alpha} -$$

$$- EJ\alpha\, y_0' \sin \alpha a \cdot \frac{1 - \cos 0{,}5\alpha a}{4EJ \cdot (0{,}5\alpha)^2} = \emptyset$$

$$y'(x_2 = a) = y_0' \cos \alpha a \cdot \cos 0{,}5\alpha a - EJ\alpha\, y_0' \sin \alpha a \frac{\sin 0{,}5\alpha a}{0{,}5\alpha \cdot 4EJ} = \emptyset$$

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$$y_0 + y_0' \frac{\sin \alpha a}{\alpha} + y_0' \cos \alpha a \cdot \frac{\sin 0{,}5\alpha a}{0{,}5\alpha} -$$

$$- y_0' \sin \alpha a \cdot \frac{1 - \cos 0{,}5\alpha a}{\alpha} = \emptyset \qquad (1)$$

$$y_0' \cos \alpha a \cdot \cos 0{,}5\alpha a - y_0' \sin \alpha a \cdot \frac{\sin 0{,}5\alpha a}{2} = \emptyset \qquad (2)$$

Iz jednadžbe (2) slijedi:

$$y_0' (2 \cos \alpha a \cdot \cos 0{,}5\alpha a - \sin \alpha a \cdot \sin 0{,}5\alpha a) = \emptyset$$

$$2 \cos \alpha a \cdot \cos 0{,}5\alpha a - \sin \alpha a \cdot \sin 0{,}5\alpha a = \emptyset$$

supstitucija: $u = \alpha a$

$$f(u) = 2 \cos u \cdot \cos 0{,}5u - \sin u \cdot \sin 0{,}5u = \emptyset$$

Metodom pokušaja: $u = 1{,}231$

$$\alpha = \frac{1{,}231}{a} = \sqrt{\frac{P_{cr}}{EJ}} \Big/^{2}$$

$$P_{cr} = \frac{1{,}52\, EJ}{a^2}$$